

## AN INTERTEMPORAL ASSET PRICING MODEL WITH STOCHASTIC CONSUMPTION AND INVESTMENT OPPORTUNITIES

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This paper derives a single-beta asset pricing model in a multi-good, continuous-time model with uncertain consumption-goods prices and uncertain investment opportunities. When no riskless asset exists, a zero-beta pricing model is derived. Asset betas are measured relative to changes in the aggregate real consumption rate, rather than relative to the market. In a single-good model, an individual's asset portfolio results in an optimal consumption rate that has the maximum possible correlation with changes in aggregate consumption. If the capital markets are unconstrained Pareto-optimal, then changes in all individuals' optimal consumption rates are shown to be perfectly correlated.

### 1. Introduction

The capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) is an important theory of the structure of equilibrium expected returns on securities in the capital markets. Empirical tests of the model have had mixed results, in that security returns do appear to be positively related to their respective measured market 'betas', but not in the precise manner implied by the CAPM.<sup>1</sup> By relaxing the assumptions involved in the derivation of the CAPM, the model has been extended to more general economies, usually at the expense of simplicity in the structure of equilibrium expected returns. This paper further develops the intertemporal extension of the CAPM that was initiated by Merton (1973) in a continuous-time model.

Merton's intertemporal CAPM with stochastic investment opportunities states that the expected excess return on any asset is given by a 'multi-beta' version of the CAPM with the number of betas being equal to one plus the number of state variables needed to describe the relevant characteristics of

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<sup>1</sup>See Jensen (1972) for a survey of many of these results.

the investment opportunity set. Since all of those state variables are not easily identified, this intertemporal extension, while quite important from a theoretical standpoint, is not very tractable for empirical testing, nor is it very useful for financial decision-making. This paper utilizes the *same* continuous-time economic framework as that used by Merton, likewise permitting stochastic investment opportunities. However, it is shown that Merton's multi-beta pricing equation can be collapsed into a single-beta equation, where the instantaneous expected excess return on any security is proportional to its 'beta' (or covariance) with respect to aggregate consumption alone. In this paper, it is also demonstrated that this result extends to a multi-good world, with an asset's beta measured relative to aggregate real consumption. The fact that this model involves a single beta relative to a specific variable, rather than many betas measured relative to unspecified variables, may make it easier to test and to implement, given certain stationarity assumptions on the joint distributions of rates of return and aggregate consumption.

Section 2 presents the continuous-time economic model with stochastic investment opportunities. General versions of the 'mutual fund' theorem of Merton (1973) and Long (1974) and of their multi-beta CAPM are briefly derived. The single-beta, single-good intertemporal CAPM as described above is derived and discussed in section 3. This derivation also generalizes a similar single-beta CAPM derived in a multi-period state preference model by Breeden and Litzenberger (1978). They derived the same pricing equation, but only for assets with cash flows that are jointly lognormally distributed with aggregate consumption. Neither consumption nor asset prices need be lognormally distributed here, but they are assumed to follow diffusion processes. A simple example is presented in section 4 to illustrate the point that the relation of an asset's return with aggregate consumption precisely measures its relevant risk, whereas the return's relation to aggregate wealth is not an adequate measure of an asset's risk.

Section 5 demonstrates that there are intertemporal analogs to the single-period results that state that all individuals' wealths will be perfectly correlated and that each individual's portfolio beta is proportional to his Pratt (1964) – Arrow (1965) measure of relative risk tolerance. In particular, it is proven that changes in all individuals' optimal consumption rates are perfectly correlated at each instant, and each individual's optimal instantaneous standard deviation of changes in consumption is proportional to his relative risk tolerance, if the capital markets permit an unconstrained Pareto-optimal allocation of consumption. For general capital markets, it is shown that each individual's optimal portfolio is such that changes in the individual's optimal consumption rate have the maximum possible correlation with changes in the aggregate consumption rate.

Section 6 presents a derivation of a 'zero-beta' intertemporal CAPM for an

economy with no riskless asset. The expected return on the zero-beta portfolio is obtained from a portfolio with returns that are uncorrelated with changes in aggregate consumption. This pricing model is an intertemporal analog to the single-period zero-beta model of Lintner (1969), Black (1972), and Vasicek (1971).

A multi-good extension of the intertemporal CAPM is presented in section 7. Long (1974) has extended Merton's multi-beta model to the multi-good case in a discrete-time economy, but this extension resulted in a pricing equation with even more terms. The focus of this section of the paper is on the derivation of a single-beta CAPM in the multi-good world. It is shown that equilibrium expected excess real returns on assets are proportional to the assets' betas with respect to aggregate *real* consumption, where aggregate real consumption is computed for an instantaneously additive price index with aggregate expenditure fractions on the various goods as weights. This result also extends the single-risk-measure asset pricing equation of Grauer and Litzenberger (1979) from a multi-good economy with strong 'homothetic' restrictions on consumption preferences to an economy with general and diverse consumption preferences. The continuous-time framework permits their covariance of an asset's return with the marginal utility of aggregate consumption to be written as a function of the asset's consumption-beta.

## 2. The economic model

The continuous-time model of this paper is very similar to the models utilized by Merton (1971, 1973), Lucas (1978), and Cox, Ingersoll and Ross (CIR) (1977). Therefore, in the interest of brevity, common facets of this model will only be sketched, with the unfamiliar reader being referred to those earlier developments of the model. Readers familiar with these continuous-time models may skip this section without losing the thrust of the paper.

Initially, it is assumed that there is a single good that may be consumed by individuals or invested via firms; a multi-good extension is presented in section 7. Individuals are assumed to behave as price takers in perfectly competitive, but possibly incomplete capital markets that are frictionless. They may trade continuously and may short-sell any assets with full use of the proceeds. Trading takes place only at equilibrium prices. Also, it is assumed that all investors have identical probability beliefs for states of the world. Individuals hold wealth in the form of risky asset shares or in an instantaneously riskless asset; the case where no riskless asset exists is presented in section 6.  $W^k$  is individual  $k$ 's wealth and  $w^k$  is his  $A \times 1$  vector of fractions of wealth invested in the various risky assets. (Throughout this paper, vectors will appear as bold italic and multi-column matrices will appear as bold roman.) Letting  $\mathbf{1}$  be a vector of ones,  $w_0^k = 1 - \mathbf{1}'w^k$  is

individual  $k$ 's fraction of wealth invested in the riskless asset. Each individual  $k$  has a stochastic number of labor units,  $y^k$ , that yield a continuous wage income rate of  $ly^k$ .<sup>2</sup>

It is assumed that there exists an  $N \times 1$  vector of state variables,  $\theta$  that (with time) describes the state of the world. For example, asset prices, dividend yields, and income rates may be written as  $P(\theta, t)$ ,  $\delta(\theta, t)$ , and  $l(\theta, t)y^k(\theta, t)$ , respectively. Assuming that the state vector  $\theta$  follows a continuous-time vector Markov process of the Ito type, the following stochastic differential equations may be written as

$$d\theta = \mu_\theta(\theta, t) dt + \sigma_\theta(\theta, t) dz_\theta, \quad (1)$$

$$\frac{dP_a}{P_a} = [\mu_a(\theta, t) - \delta_a(\theta, t)] dt + \sigma_a(\theta, t) dz_a \quad \text{for each asset } a, \quad (2)$$

$$dy^k = \mu_{yk}(\theta, t) dt + \sigma_{yk}(\theta, t) dz_{yk} \quad \text{for each individual } k, \quad (3)$$

where the drift and diffusion coefficients in (2) and (3) may be obtained from those in (1) by Ito's Lemma.<sup>3</sup> Throughout,  $\mu_j(\theta, t)$  represents the expected rate of change in variable  $j$  at time  $t$ , when the state vector is  $\theta$  at that time. Similarly,  $\sigma_j(\theta, t)$  represents the standard deviation of that rate of change, which depends upon time and the state vector;  $\sigma_\theta$  is the diagonal matrix of the instantaneous standard deviations of the state variables. The  $z_\theta$  variables are correlated Weiner processes, having zero means, unit variances per unit of time, and variance-covariance matrix and correlation matrix  $V_{\theta\theta}$ , which may depend upon  $\theta$  and  $t$ .

Although there are a number of technical conditions that functions of Ito processes must meet for the application of Ito's Lemma [and for the representations of (2) and (3) to be rigorous], the economic restrictions on the movement of asset prices and incomes are not severe. Asset prices, dividends and incomes must follow continuous sample paths, but their levels, their mean rates of change and their variances and covariances may be stochastic, depending upon the evolution of the state vector over time. Thus,

<sup>2</sup>The labor-leisure choice is not examined in this paper. The formal model of sections 2-6 treats an individual's labor units supplied as stochastic and possibly correlated with all other economic variables, but there is no disutility for labor supplied. The multi-good model of section 7 could be adapted to handle the labor-leisure choice.

<sup>3</sup>For discussions of stochastic differential equations and of Ito's Lemma, see Merton (1978), Arnold (1974, sec. 5.3-5.5), Gihman and Skorohod (1972, part II, ch. 2, sec. 6), Kushner (1967, sec. 1.4), or McKean (1969, ch. 2). For discussions of the optimal control of these stochastic processes, see Arnold (1974, sec. 13.1-13.2) or Kushner (1967, ch. 6). For less technical discussions of these processes and theorems and for applications of them in economic models, see Merton (1971, 1973), Garman (1976), and Cox, Ingersoll and Ross (1977).

in Merton's (1973) terminology, the 'investment opportunity set' may be stochastic here. The state variables need not be restricted in number, nor do they need to be specified for the purposes of this paper. Restrictions on their number would restrict the dimensionality of the price system, as noted by Rosenberg and Ohlson (1976).

For the derivations that follow, it is not necessary to explicitly examine firms' production decisions and the supply of asset shares, provided that the assumptions made are consistent with optimal behavior of firms in a general equilibrium model. To be consistent with general equilibrium, prices must be recognized to be endogenously determined through the equilibrium of supply and demand. The model presented is consistent with endogenously determined prices if, as assumed, all random shocks to the economy are captured as elements of the state vector,  $\theta$ . These random shocks may affect both the supplies and demands for shares. However, assuming that both supply and demand functions are functions of the state variables (shocks) that follow Ito processes, the equilibrium prices that arise will also follow Ito processes that are representable as in (2). This statement follows from Ito's Lemma, subject to the qualification that the supply and demand functions be sufficiently smooth for Ito's Lemma to apply.<sup>4</sup> Thus, the economic model of (1)–(3) is consistent with endogenously-determined prices.

As an example of a supply side that can be imbedded in this model without changing any of the analysis, consider the following economy. The output of the economy is produced by  $F$  different productive units (firms) under conditions of uncertainty about current investment productivity and about future investment technology. Firms buy stocks of the good and rent labor units of the good for use in their production processes. The current stock of the good that firm  $f$  owns is  $x_f$ , and the current amount of labor employed by it is  $y_f$ . The  $F \times 1$  vectors of capital investment and labor employment by the various firms are denoted  $\mathbf{x}$  and  $\mathbf{y}$ , respectively, and the current wage rate is  $l$ . Changes in the amount of the good that a firm has are caused by its production, less its wage payments and dividends,  $d_f$ , and plus any new capital infusions,  $\eta_f$ , from sales of stock (negative  $\eta_f$  represents stock repurchases). It is assumed that such changes in the stock of the good that productive units have may be described by a system of stochastic differential equations of the Ito type,

$$d\mathbf{x} = [\boldsymbol{\mu}_x(\mathbf{x}, \mathbf{y}, e, t) - l\mathbf{y} - \mathbf{d} + \boldsymbol{\eta}] dt + \boldsymbol{\sigma}_x(\mathbf{x}, \mathbf{y}, e, t) d\mathbf{z}_x, \quad (4)$$

and

$$de = \boldsymbol{\mu}_e(e, t) dt + \boldsymbol{\sigma}_e(e, t) d\mathbf{z}_e, \quad (5)$$

<sup>4</sup>See footnote 3.

where  $e$  is an  $E \times 1$  vector of indices that describe the productivity of current technology, and  $z_x$  and  $z_e$  are vectors of correlated Wiener processes. The vector of expected production rates is  $\mu_x$ , and  $\sigma_x$  is the diagonal matrix of instantaneous standard deviations of the various production rates. Both expected and random components of a firm's production may depend upon the capital and labor employed and upon the current level of technology. As indicated by (5), technological change is assumed to be random, with the productivity indices following a vector Markov process. Although it is not done here, it is also possible (with more notation) to model expenditures on research and development that would affect the rates of technological change in the various production processes.

In the example, each firm may issue a number of different securities, such as debt and equity, that contractually partition its cash flows over time among investors in the firm. Each firm is assumed to maximize the value of its securities, net of input costs. For slightly greater generality, it is also assumed that individuals may issue or purchase a number of contractually defined securities ('side bets') that have zero net supplies. Options and forward contracts are permissible in this class of financial assets.

In a rational expectations equilibrium, asset prices in this economy are functions of the consumption preferences of individuals and time, which are non-stochastic, and the following stochastic variables: (1) the current productivities of the production processes, (2) the current supplies of capital and labor, and (3) the current distribution of income and wealth among individuals. Since all of these stochastic variables follow Ito processes in this model, and since they jointly comprise a Markov system, the initial representation of prices, dividends and income rates as functions of a Markov vector of state variables and time, as given by eqs. (1)–(3), is consistent with the existence of a production sector as sketched and with the endogenous determination of asset prices. Changes in the state vector,  $\theta$ , for this economy are the results of stochastic production and stochastic technological change, which are the underlying exogenous variables of this example. Any other economies with supply side structures that are consistent with (1)–(3), given the preference and the other assumptions, are also governed by the theorems and pricing relations of this paper.

It is possible, with certain preference and/or probabilistic assumptions, that fluctuations in some of the elements of the state vector  $\theta$  do not affect any individual's expected utility of lifetime consumption, given the individual's wealth. For example, certain elements of the state vector may affect the distribution of payoffs between two assets in such a way that the total payoff to the two is unaffected. If all individuals hold identical fractions of the two assets, then their expected utilities are unaffected by fluctuations in those state variables, assuming that the state variables have no other effects. To distinguish between state variables that do affect at least one individual's

expected utility, given the person's wealth, it is convenient to define another state vector,  $s$ , that contains those state variables that do affect at least one individual's expected utility, given his wealth. This  $S \times 1$  vector of variables is a subset of the comprehensive state vector,  $\theta$ , and is assumed to follow a vector Markov process. Summarizing, each individual's expected utility of (remaining) lifetime consumption may be written as a function of his wealth, the vector of relevant state variables, and time,  $J^k = J^k(W^k, s, t)$ , where

$$ds = \mu_s(s, t) dt + \sigma_s dz_s. \tag{1'}$$

As all of the subsequent analysis and theorems are in terms of only these state variables,  $s$ , they are referred to throughout the paper as the 'state vector' or as the 'vector of state variables'.

Each individual  $k$  is assumed to maximize the expected value at each instant in time of a time-additive and state-independent von Neumann-Morgenstern utility function for lifetime consumption,

$$E_t \left\{ \int_t^{t^k} U^k(c^k(\tau), \tau) d\tau + B^k[W^k(t^k), t^k, s(t^k)] \right\}, \tag{6}$$

where  $t^k$  is individual  $k$ 's time of death, and  $U^k$  and  $B^k$  are his strictly quasiconcave utility and bequest functions of consumption,  $c^k$ , and terminal wealth,  $W^k(t^k)$ , respectively.<sup>5</sup>  $E_t$  is the expectation operator at time  $t$ , conditional upon the state of the world at that time.

At each instant, individual  $k$  chooses an optimal rate of consumption,  $c^k$ , and an optimal portfolio of risky assets,  $w^k W^k$ . Given these choices, Merton (1971) has shown that the individual's wealth will follow the stochastic differential equation,

$$dW^k = [w^{k'}(\mu_a - r) + r]W^k dt + (ly^k - c^k) dt + W^k w^{k'} \sigma_a (dz_a), \tag{7}$$

where  $r$  is the instantaneously risk-free interest rate,  $r = r \cdot \mathbf{1}$ ,  $\mu_a$  is the  $A \times 1$  vector of expected total (capital gains and dividends) rates of return on assets, and  $\sigma_a$  is the  $A \times A$  diagonal matrix of assets' instantaneous standard deviations. Thus,  $\mu_a$ ,  $\sigma_a$ , and  $dz_a$  are all as presented in (2).

Let  $J^k(W^k, s, t)$  be the maximum expected utility of lifetime consumption in (6) that is obtainable with wealth  $W^k$  and opportunities  $s$  at time  $t$ . Under certain conditions, if there exists a well-behaved function  $J^k(W^k, s, t)$  and controls  $c^k(W^k, s, t)$  and  $w^k(W^k, s, t)$  that solve the following problem subject

<sup>5</sup>Under certain conditions, individuals' lifetimes may be uncertain. See Merton (1973) or Richard (1975). See the Richard paper for an analysis of optimal life insurance rules in a continuous-time model.

to the constraint of (7), then the consumption and portfolio decisions are optimal (with superscript  $k$  suppressed),<sup>6</sup>

$$0 = \max_{\{c, w\}} \left\{ U(c, t) + (J_w J_s J_r) \begin{pmatrix} [w'(\mu_a - r) + r]W + ly - c \\ \mu_s \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} V_{ww} & V_{ws} \\ V_{sw} & V_{ss} \end{pmatrix} \square \begin{pmatrix} J_{ww} & J_{ws} \\ J_{sw} & J_{ss} \end{pmatrix} \right\}, \tag{8}$$

where subscripts of the  $J$  function represent partial derivatives with respect to wealth ( $J_w$ ) and the various state variables ( $J_s$ ). The matrix of  $V$ 's is a partitioning of the variance-covariance matrix of the individual's wealth and the state variables. The box multiply sign implies that corresponding elements of the two matrices are multiplied, then summed. Note that the individual's variance rate for wealth is  $V_{ww}^k = (W^k)^2 w^k V_{aa} w^k$ , and his vector of covariances of wealth with the state variables is  $V_{ws}^k = W^k w^k V_{as}$ , where  $V_{aa}$  is the  $A \times A$  variance-covariance matrix of asset returns, and  $V_{as}$  is the  $A \times S$  matrix of covariances of asset returns with state variables.

First-order conditions for an interior maximum in (8) may be stated as

$$U_c^k(c^k, t) = J_w^k(W^k, s, t), \tag{9}$$

and

$$w^k W^k = (-J_w^k / J_{ww}^k) V_{aa}^{-1} (\mu_a - r) - V_{aa}^{-1} V_{as} (J_{sw}^k / J_{ww}^k). \tag{10}$$

These conditions give the individual's optimal risky asset portfolio, (10), and state that the marginal utility of another unit of consumption must equal the indirect marginal utility of wealth for an optimal policy.

The following portfolio allocation theorem is obtained directly from individuals' portfolio demands as given by (10). Its proof is in appendix 1.

*Theorem 1. S+2 Funds. All individuals in this economy, regardless of preferences, may obtain their optimal portfolio positions by investing in at most S+2 funds. These funds may be chosen to be: (1) the instantaneously riskless asset, (2) the S portfolios having the highest correlations, respectively, with the S state variables summarizing investment and income opportunities, and (3) the market portfolio.*

Of course, any  $S+2$  funds that span the same vector space would also suffice.

<sup>6</sup>See footnote 3 for references for this result and the conditions under which it is valid.



To see that Merton's (1973) 'multi-beta' asset pricing model obtains in this economy when betas are measured with respect to aggregate wealth and the returns of assets that hedge against changes in the various state variables, aggregate individuals' portfolio demands in (10) and substitute in equilibrium expected excess returns for the market portfolio,  $(\mu_M - r)$ , and for assets perfectly correlated with the state variables,  $(\mu_s - r)$ , assuming that such assets exist. Doing this, Merton's model is obtained,<sup>7</sup>

$$\mu_a - r = \beta_{a, Ms} \begin{pmatrix} \mu_M - r \\ \mu_s^* - r \end{pmatrix}, \tag{11}$$

where  $\beta_{a, Ms}$  is the  $A \times (S+1)$  matrix of 'multiple-regression' betas for all assets on the market and on the assets perfectly correlated with the state variables. This type of multi-beta equation was also derived in a discrete-time model by Long (1974). As both Merton and Long noted, the Sharpe-Lintner CAPM will not generally hold in these intertemporal economic models – expected excess returns are not proportional to market betas in these models with stochastic investment opportunities.

As shown by Garman (1976) and by Cox, Ingersoll and Ross (1977), each asset's price in this economy is a solution to a second-order partial differential equation in its price. This 'fundamental valuation equation' may be obtained for any asset by using Ito's Lemma to find its expected

<sup>7</sup>When assets with returns that are perfectly correlated with the state variables do not exist, a multi-beta CAPM as in (11) holds with one modification: expected excess returns for the  $S$  portfolios in (11) are those of the  $S$  portfolios with the maximum correlations with the  $S$  state variables, respectively, which have portfolio weights that are proportional to the columns of  $V_{aa}^{-1}V_{as}$ . Similarly, the  $S$  non-market betas required for each asset for (11) may be measured relative to the returns on these  $S$  most highly correlated portfolios. Briefly, the proof is as follows. From Appendix 1, (A.1),

$$w^M = k_1 V_{aa}^{-1}(\mu_a - r) + w_s k_2,$$

where  $w_s$  is the  $A \times S$  matrix of  $S$  portfolios with the maximum correlation of returns with the various state variables, respectively. Pre-multiplying this equation by  $V_{aa}$  gives

$$\begin{aligned} \mu_a - r &= V_{aM}(1/k_1) + V_{as}(-k_2/k_1) = V_{a, Ms} \begin{pmatrix} 1/k_1 \\ -k_2/k_1 \end{pmatrix}, \\ \begin{pmatrix} \mu_M - r \\ \mu_s - r \end{pmatrix} &= V_{Ms, Ms} \begin{pmatrix} 1/k_1 \\ -k_2/k_1 \end{pmatrix}, \end{aligned}$$

where  $V_{Ms, Ms}$  is the  $(S+1) \times (S+1)$  variance-covariance matrix for the market and the most correlated portfolios' returns. Substituting this result into the previous equation gives

$$\mu_a - r = V_{a, Ms} V_{Ms, Ms}^{-1} \begin{pmatrix} \mu_M - r \\ \mu_s - r \end{pmatrix} = \beta_{a, Ms} \begin{pmatrix} \mu_M - r \\ \mu_s - r \end{pmatrix},$$

which is the result stated.

instantaneous return from the function  $P(\theta, t)$  and then by equating this drift rate to the equilibrium drift rates implied by the multi-beta model of (11). An interpretation of the general mathematical solution to the valuation equation is given in the CIR paper, but useful closed-form solutions are known only for a few assets, and then only under highly restrictive preference and state assumptions. In particular, CIR assume logarithmic utility functions and a single state variable to derive their closed-form solution for the term structure of interest rates. In this paper, no restrictions on state variables are imposed, and only the relatively weak preference assumption of (6) is made. Consequently, the goal here is to simplify the expression relating asset risks and returns, rather than to solve for an explicit pricing function,  $P(\theta, t)$ . The next section demonstrates that the multi-beta intertemporal CAPM of (11) can be collapsed into a single-beta intertemporal CAPM, with no additional assumptions.

### 3. A 'single-beta' intertemporal asset pricing model

Up to this point, the consumption–investment analysis is virtually the same as in Merton's (1973) continuous-time development, but with slightly more discussion of the supply side. An individual's portfolio holdings are found in terms of his indirect utility function for wealth,  $J^k(W^k, s, t)$ , and equilibrium expected asset returns are correspondingly found in terms of aggregate wealth and the returns on assets that are perfectly correlated with changes in the various state variables, if they exist. This paper focuses upon the individual's direct utility function for consumption,  $U^k(c^k, t)$ , in the analysis of equilibrium expected returns on assets. The two approaches are intimately linked by the optimality condition that the marginal utility of consumption equals the marginal utility of wealth.

To restate the optimal portfolio demands in terms of the individual  $k$ 's optimal consumption function,  $c^k(W^k, s, t)$ , note from (9) that:  $J_W^k = U_c^k$ , which implies that  $J_{W_s}^k = U_{cc}^k c_s^k$  and  $J_{W W}^k = U_{cc}^k c_W^k$ , where subscripts of  $U$ ,  $J$  and  $c$  denote partial derivatives. Define  $T^k$  to be individual  $k$ 's absolute risk tolerance:  $T^k = -U_c^k / U_{cc}^k$ . Then the optimal portfolio may be written as

$$w^k W^k = (T^k / c_W^k) V_{aa}^{-1} (\mu_a - r) - V_{aa}^{-1} V_{as} (c_s^k / c_W^k), \quad (12)$$

where  $\mu_a - r$  is the vector of instantaneous expected excess returns on assets,  $V_{aa}$  is their variance–covariance matrix, and  $V_{as}$  is the  $A \times S$  matrix of covariances of asset returns with changes in the state variables.

Pre-multiplying (12) by  $c_W^k V_{aa}$  and rearranging terms gives

$$T^k (\mu_a - r) = V_{aW} c_W^k + V_{as} c_s^k, \quad (13)$$

where  $V_{aW^k}$  is the vector of covariances of asset returns with  $k$ 's wealth change. Since  $k$ 's optimal consumption is a function,  $c^k(W^k, s, t)$ , of his wealth, the state variables and time, Ito's Lemma implies that the local covariances of asset returns with changes in  $k$ 's consumption rate are given by

$$V_{ac^k} = V_{aW^k} c_{W^k}^k + V_{as} c_{s_i}^k, \quad (14)$$

which is the right-hand side of (13). Intuitively, (14) can also be seen by noting that the random change in  $k$ 's consumption rate is locally linear in the random changes in  $k$ 's wealth and the state variables, with the weights in the linear relation being the partial derivatives of  $k$ 's consumption with respect to wealth and the state variables. Thus, the local covariance of asset  $j$ 's return with  $k$ 's change in consumption is

$$\begin{aligned} \text{cov}(\tilde{r}_j, d\tilde{c}^k) &= \text{cov}\left(\tilde{r}_j, c_{W^k}^k(d\tilde{W}^k) + \sum_i c_{s_i}^k(d\tilde{s}_i)\right) \\ &= c_{W^k}^k \text{cov}(\tilde{r}_j, d\tilde{W}^k) + \sum_i c_{s_i}^k \text{cov}(\tilde{r}_j, d\tilde{s}_i), \end{aligned} \quad (15)$$

which is what is stated by (14).

By substituting (14) into (13), it is seen that each individual will choose an optimal portfolio in such a way that the local covariance of each asset's return with changes in his optimal consumption is proportional to the asset's expected excess return,

$$V_{ac^k} = T^k(\mu_a - r). \quad (16)$$

This relation holds for each individual  $k$  and can be aggregated by summing over all individuals in (16). Using the aggregate relation, defining the aggregate consumption rate to be  $C$ , and defining a measure of aggregate risk tolerance to be  $T^M = \sum_k T^k$ , it follows that the expected excess returns on assets in equilibrium will be proportional to their covariances with changes in aggregate consumption,

$$\mu_a - r = (T^M)^{-1} V_{aC}. \quad (17)$$

By dividing both the random consumption change and aggregate risk tolerance by current aggregate consumption, (17) may be expressed in terms of aggregate relative risk tolerance and return covariances with changes in the logarithm of consumption (percentage rates of change of consumption),

$$\mu_a - r = (T^M/C)^{-1} V_{a, \ln C}. \quad (17')$$

For any portfolio  $M$  with weights  $w^M$ , pre-multiplying (17') by those weights gives

$$(\mu_M - r)/\sigma_{M, \ln C} = (T^M/C)^{-1}, \quad (18)$$

and

$$\begin{aligned} \mu_a - r &= (V_{a, \ln C}/\sigma_{m, \ln C})(\mu_M - r) \\ &= (\beta_{aC}/\beta_{MC})(\mu_M - r), \end{aligned} \quad (19)$$

where  $\beta_{aC}$  and  $\beta_{MC}$  are the 'consumption-betas' of asset returns and of portfolio  $M$ 's return. The consumption-beta for any asset  $j$ 's return is defined to be

$$\beta_{jC} = \text{cov}(\tilde{r}_j, d \ln \tilde{C}) / \text{var}(d \ln \tilde{C}). \quad (20)$$

If there exists a security whose return is perfectly correlated with changes in aggregate consumption over the next instant, then the risk-return relation of (19) can be written in terms of assets' betas measured relative to that security's return,  $\beta_C$ , and the expected excess return on this security,  $\mu_C^* - r$ ,<sup>8</sup>

$$\mu_a - r = \beta_C(\mu_C^* - r). \quad (21)$$

Portfolio  $M$  may be any measure of the market portfolio or any other portfolio. Eq. (19) states that the ratio of expected excess returns on any two assets or portfolios in equilibrium will be equal to the ratio of their betas measured relative to aggregate consumption. Thus, the relevant risk of a security's return may be summarized by a single beta with respect to consumption – a considerable simplification over the Merton multi-beta derivation, at no loss of generality in assumptions.

The intertemporal asset pricing relation of (19) or (21) holds at each instant in time, but does not necessarily hold for returns and betas that are measured over finite periods of time. Breeden and Litzenberger (1978) have shown that assumptions of identical constant relative risk aversion utility functions for individuals and lognormally distributed consumption are sufficient to derive (19) for returns and consumption-betas measured over finite time periods.

There are two ways to understand the economic intuition of this result:

<sup>8</sup>In general, even if there does not exist a portfolio whose return is perfectly correlated with aggregate consumption, the consumption-betas in (19), (20), and (21) may be equivalently derived as the betas measured relative to the returns on the asset portfolio that has the most highly correlated returns with changes in aggregate consumption. The proof is a univariate version of footnote 7, working from (17) and the fact that the most correlated portfolio has weights proportional to  $V_{aa}^{-1}V_{aC}$ .

the first focuses upon the marginal rates of substitution between consumption today and consumption in the future, whereas the second interpretation focuses upon the level of wealth and the productivity of investments at future dates and states. Both explanations are briefly presented here. Although capital market completeness was not necessary for the ICAPM of (19), the first explanation is cast in the simplified framework of complete markets.<sup>9</sup>

Any asset may be described for valuation purposes by its total payoff, price and dividend, in the various possible states of the world in the next instant (a period in discrete time). The value in equilibrium of a \$1 payoff in a particular state of the world at a future date is equal to the state's probability multiplied by the ratio of the marginal utility of consumption at the future state to the marginal utility of consumption in the current period. That is,

$$\lambda_{t_1 s_1} = \pi_{t_1 s_1} (U_c^k(c_{t_1 s_1}^k, t_1) / U_c^k(c_t^k, t)) \quad \text{for all } k, \quad (22)$$

where  $\lambda_{t_1 s_1}$  is  $k$ 's shadow price at time  $t$  of \$1 received at time  $t_1$  if state  $s_1$  occurs,  $\pi_{t_1 s_1}$  is the probability of that state,  $c_{t_1 s_1}^k$  is  $k$ 's optimal consumption if that state occurs, and  $c_t^k$  is  $k$ 's current consumption. The value of any asset having a dividend,  $d_{t+1, s}$ , and price,  $P_{t+1, s}$ , at time  $t+1$  in different states of the world is

$$P_t = \sum_s (d_{t+1, s} + P_{t+1, s}) \lambda_{t+1, s}, \quad (23)$$

which must be the same for all individuals behaving optimally. Thus, the price per unit of probability for these elementary state-contingent claims varies among states only as planned consumption varies among states. The relation is inverse between planned consumption and the price/probability ratio for the state, due to the diminishing marginal utility of consumption. Therefore, holding the expected payoff on an asset constant, the value of the asset will be negatively related to its covariance with the individual's consumption. As seen from (22), for each date in this economy, if the capital markets are Pareto-optimal, the larger  $\lambda_{t_1 s_1} / \pi_{t_1 s_1}$  is, the smaller each individual's consumption is in state  $s_1$ .<sup>10</sup> Since each individual's planned consumption in various states is positively and monotonically related to aggregate planned consumption, it can also be said that, holding the

<sup>9</sup>Theorem 3 of section 6 characterizes the relation of individuals' optimal consumption rates to the aggregate consumption rate for the general case of incomplete capital markets. Following that theorem, additional discussion of the pricing results of (19) and (21) is presented.

<sup>10</sup>For a detailed analysis of optimal consumption allocations in a multi-period state preference framework, see Breeden and Litzenberger (1978)

expected payoff on an asset constant, the value of the asset will be negatively related to its covariance with aggregate consumption. This implies relatively large (small) equilibrium expected returns on assets with relatively large (small) covariances with aggregate consumption, as is indicated by (19).

The key to this analysis is the relation between low levels of aggregate consumption and highly-valued state payoffs via the relation between value and marginal rates of substitution of consumption. The reason that payoff covariances with more distant levels of aggregate consumption (or their present value, aggregate wealth) do not appear explicitly in the pricing equation is that they are already reflected in the levels of equilibrium asset prices that will occur in alternative states at the next instant. That is, the asset's value in the next period appropriately reflects the covariances of its more distant payoffs with more distant levels of aggregate consumption.

The alternative, equivalent explanation is presented somewhat less rigorously, but may be more intuitive in light of the development of the finance literature. Holding expected payoffs constant between two assets, one asset's payoff probability distribution is preferred to the other's, if it tends to pay more highly in states where another dollar to invest gives large benefits (high marginal utility) and tends to pay relatively less in states where another dollar invested gives small benefits (low marginal utility). Whether an additional dollar invested is more or less beneficial depends upon: (1) the wealth of the economy in that state, via the diminishing marginal utility of wealth (future consumption), and (2) the physical productivity of investments in the state, that is, the marginal rate of transformation of goods today into goods in the future. The diminishing marginal utility of wealth was the driving force for the single-period CAPM and its portfolio diversification theorem. In the intertemporal model, as Merton (1973) has shown, changing investment opportunities create what he terms 'hedging demands' for assets, with their concomitant implications for equilibrium expected returns on assets.

An asset's covariance with aggregate consumption is all that is necessary for asset pricing, because aggregate consumption is perfectly negatively correlated with the marginal utility of an additional dollar of wealth invested through the optimality condition:  $U_c(c, t) = J_w(W, s, t)$ . Holding investment opportunities constant, if wealth is relatively high in a state, then the value per dollar of payoffs in that state is low. Optimal consumption is relatively high in that state. Holding wealth constant, if investment opportunities are relatively good in a state, then the present value of a dollar payoff in that state is high, as it can be invested quite profitably. In this case, optimal consumption is relatively low for individuals. *Always*, when the value of an additional dollar payoff in a state is high, consumption is low in that state, and when the value of additional investment is low, optimal consumption is high. This is *not* always true for wealth, when investment opportunities are

uncertain. It is quite possible that there are states of the world where wealth is high and, yet, the marginal utility of a dollar is high due to the excellent investment opportunities in the state. Similarly, it is quite possible that there are states where wealth is low and, yet, the marginal utility of a dollar is low due to poor investment opportunities. Given preferences, wealth is not a sufficient statistic for the marginal utility of a dollar – consumption is.<sup>11</sup> For optimum consumption and portfolio choices, an individual's marginal utility of wealth or consumption is a monotonically decreasing function of consumption. For this reason, holding the expected payoff on an asset constant, its present value is a decreasing function of its covariance with aggregate consumption. Consequently, the higher that an asset's beta with respect to consumption is, the higher its equilibrium expected rate of return.

Note that this analysis is consistent with the derivations of the market-oriented CAPM by Sharpe (1964) and Lintner (1965) in a single-period context, and by Merton (1973) and Long (1974) in an intertemporal model. In the single-period model, all wealth is consumed at the end of the period, so investment opportunities are irrelevant. In Merton's model, investment opportunities are required to be constant for the derivation of the single-beta CAPM; thus, wealth is a sufficient statistic for marginal utility in that model. Merton and Long's multi-beta pricing models are derived with stochastic investment opportunities, as in this paper; the foregoing analysis demonstrates that wealth is not a sufficient statistic for marginal utility in their models.

#### **4. An example**

A simple example more graphically illustrates the main point. Consider a 3-date economy with many identical individuals and a single good called wheat. The current stock of wheat is the entire wealth of the economy. At each date, the amount of wheat to be consumed and the (residual) amount to be invested must be determined; wheat invested produces more wheat that will be available for future consumption. Assume that the optimal consumption/investment decision has already been made for date 1 and that the amount of wheat available for consumption and investment at date 2 will either be 200 bushels/person or 231 bushels/person, depending upon the state of the world. Furthermore, assume that the physical productivity of wheat invested at date 2 for consumption at date 3 may either be 0% or 20%, depending upon the state of the world. This 're-investment rate' will be

<sup>11</sup>The fact that consumption is a sufficient statistic for an individual's marginal utility is due to the assumption that individuals have time-additive and state-independent preferences for consumption.

known for certain at date 2, but is unknown at date 1. Constant returns to scale are assumed.

At date 2, each individual chooses consumption,  $c_2$ , and investment,  $W_2 - c_2$ , which results in consumption at date 3 of  $c_3 = (W_2 - c_2)(1 + r_2)$ , where  $r_2$  is the physical productivity of investment at date 2. Each individual's utility function is  $u(c_2, c_3) = c_2^{0.5} + c_3^{0.5}$ . It may be verified that the optimal consumption at date 2 is  $c_2 = W_2/(2 + r)$ . At date 2 there are four possible states of the world, representing the different possible combinations of wealth and productivity,  $W_2$  and  $r$ , respectively. Consumption and the marginal utility of another bushel of wheat at date 2 for either consumption or investment will depend upon the state of the world as shown in table 1.

It is seen from table 1 that marginal utility tends to be negatively related to wealth, but not perfectly. In particular, note that wealth in state 4 is greater than that in state 1, but marginal utility in state 4 is higher than that in state 1. This is true because the difference in physical productivity between the two states has offset the decline in marginal utility caused by the wealth differential. Since marginal utilities at time 2 are essential in the determination of prices of assets at time 1 from their state-contingent payoffs, covariances with wealth are inadequate risk measures, even in a mean-variance model.

Table 1  
Consumption, wealth, and marginal utility: An example.

State	Wealth	Physical productivity	Optimal consumption	Marginal utility
1	220	0%	110	0.0476
2	220	20%	100	0.0500
3	231	0%	115.5	0.0465
4	231	20%	105	0.0488

From table 1, it is seen that consumption is perfectly negatively related to marginal utility, as it must be with state-independent preferences. As a consequence, in the locally mean-variance, continuous-time model, covariance with consumption is the relevant risk measure for the pricing of assets.

## 5. Properties of individuals' optimal consumption functions

In the single-period portfolio theory of Markowitz (1952), Sharpe (1964), Lintner (1965) and Mossin (1966), two important results were obtained: (1) all individuals hold the same risky asset portfolio or, alternatively stated, all



individuals' rates of return on wealth are perfectly positively correlated, and (2) each individual's optimal portfolio beta or portfolio standard deviation is proportional to his Pratt (1964) – Arrow (1965) measure of relative risk tolerance. Clearly, from the portfolio theory of section 2, neither of these results holds in the intertemporal choice model with stochastic investment opportunities. This section presents two analogous results that do obtain in the intertemporal model, if the capital markets permit an unconstrained Pareto-optimal allocation: (1) at any instant, the changes in all individuals' optimal consumption rates are perfectly positively correlated, and (2) at each instant, every individual's instantaneous standard deviation of changes in his consumption rate is proportional to his Pratt–Arrow measure of relative risk tolerance.<sup>12</sup>

The first result, which was discussed in section 3, is stated more precisely by the following theorem:

*Theorem 2. Optimal Consumption Paths.* *Given the continuous-time economic model and the assumption that the capital markets permit an unconstrained Pareto-optimal allocation of consumption, at every instant in time, the change in each individual's optimal consumption rate is perfectly positively correlated with the change in every other individual's optimal consumption rate and with the change in the aggregate consumption rate for the economy.*

*Proof.* The assumption of Pareto-optimal capital markets implies that the state-contingent allocation of consumption is the same as when there exists, at each instant in time, the market portfolio, a riskless asset, and a set of portfolios whose returns are perfectly correlated with the various state variables that affect individuals' optimal consumption rates,  $c^k(W^k, s, t)$ .<sup>13</sup> As shown in section 2, individuals would need only to trade in those assets to achieve their optimal portfolios. Letting  $\mu$  and  $V$  represent the  $(S+1) \times 1$  drift vector and the  $(S+1) \times (S+1)$  incremental covariance matrix for the market portfolio and those  $S$  portfolios' rates of return, respectively, it is shown in appendix 2 that the instantaneous covariance between individual  $k$ 's changes in consumption and individual  $j$ 's is

$$\text{cov}(c^k, c^j) = T^k T^j (\mu - r)' V^{-1} (\mu - r). \quad (24)$$

From (24) letting  $\gamma = (\mu - r)' V^{-1} (\mu - r)$ , the correlation between  $k$ 's and  $j$ 's

<sup>12</sup>The relative risk tolerance referred to is calculated from the individual's (direct) utility function for consumption; it is not necessarily equal to the individual's risk tolerance measured by his (indirect) utility function for wealth. The 'direct' measure does not depend upon the state of the world, given the individual's consumption, whereas the 'indirect' measure in general does depend upon the state vector, given wealth.

<sup>13</sup>For a proof, see Breeden (1977, ch. 5).

changes in consumption is (where 'std' represents an instantaneous standard deviation)

$$\text{corr}(c^k, c^j) = \frac{\text{cov}(c^k, c^j)}{\text{std}(c^k)\text{std}(c^j)} = \frac{T^k T^j \gamma}{\sqrt{(T^k)^2 \gamma} \sqrt{(T^j)^2 \gamma}} = 1. \quad (25)$$

Similarly, by aggregating in (24) each individual's correlation of consumption with the aggregate is seen to be unity. Q.E.D.

Theorem 2 could have been anticipated by noting that Breeden and Litzenberger (1978) proved that an individual's optimal consumption at any date in the multiperiod economy may be expressed as a function of only aggregate consumption at that date. They utilized an assumption of partial homogeneity in beliefs and they assumed that individuals' preferences for consumption were time-additive and state-independent, as is assumed in section 2, eq. (6). With homogeneous beliefs as assumed here, the functional relationship between each individual's consumption rate and the aggregate consumption rate is strictly monotonic and increasing. Given their results, Ito's Lemma provides Theorem 2 in the continuous-time economy, since by Ito's Lemma any random variable that follows an Ito process is (locally) perfectly positively correlated with any positive, strictly monotonic function of it.

To see that risk tolerance (or, inversely, risk aversion) is reflected proportionally in each individual's standard deviation of changes in his optimal consumption path, note that from (24):  $\text{std}(c^k) = T^k \sqrt{\gamma}$  and that  $\text{std}(c^k)/\text{std}(C) = T^k/T^M$ . Similarly, in terms of standard deviations of growth rates,  $\text{std}(\ln c^k)/\text{std}(\ln C) = T^{**k}/T^{**M}$ , where  $T^{**k} = T^k/c^k$  is  $k$ 's relative risk tolerance and  $T^{**M} = T^M/C$  is an aggregate measure of relative risk tolerance. The implication is intuitive: those who are very risk averse will choose consumption paths with low variability, compared to those chosen by individuals who are less risk averse. Of course, in the limiting case of an individual with infinite risk aversion, the individual would choose complete insurance against any fluctuation in his consumption path. His wealth would be variable in such a way as to offset any impact of changing investment opportunities on his optimal consumption. In general, those who are more or less risk averse than average can be identified by empirically observing the standard deviations of individuals' consumption rates. They cannot be identified merely from their asset portfolios, as was implied by single-period portfolio theory.

In general capital markets, it may or may not be possible to achieve an unconstrained Pareto-optimal allocation of consumption with portfolios of available securities. For example, this situation may occur if there does not exist a portfolio of assets with a return that perfectly 'hedges' (in Merton's

terminology) against changes in one of the state variables. When an unconstrained Pareto-optimal allocation is not possible, one assumption of Theorem 2 is violated and changes in individuals' optimal consumption rates are not necessarily perfectly correlated with each other or with changes in the aggregate consumption rate. In a general capital market, the following theorem holds:

*Theorem 3. Consumption Allocations in General Capital Markets. Given a continuous-time economic model with general capital markets, at every instant in time, the optimal portfolio for each individual results in changes in the individual's optimal consumption rate that have the maximum possible correlation with changes in the aggregate consumption rate.*

*Proof.* Individual  $k$ 's optimal portfolio given in (12) maximizes the covariance of  $k$ 's changes in consumption with changes in aggregate consumption for a given variance of  $k$ 's changes in consumption. That is:  $w^k W^k$  solves

$$\begin{aligned} \max_w & c_w^k w' V_{aC} + c_s^k V_{sC} \\ & + \lambda \left[ (c_w^k w' c_s^k) \begin{pmatrix} V_{aa} & V_{as} \\ V_{sa} & V_{ss} \end{pmatrix} \begin{pmatrix} c_w^k w \\ c_s^k \end{pmatrix} - \text{var}(dc^k) \right]. \end{aligned}$$

By maximizing  $k$ 's covariance of consumption changes with changes in aggregate consumption, for given variances of  $k$ 's consumption changes and aggregate consumption changes, the correlation coefficient between the individual's consumption and aggregate consumption is maximized. Q.E.D.

This result provides an explanation for the fact that the derivation of the intertemporal CAPM does not require Pareto-optimal capital markets, i.e., the fact that perfect hedges against changes in all of the state variables are not necessary for the derivation. Since each individual's optimal portfolio maximizes the correlation of his consumption with aggregate consumption, fluctuations in each individual's consumption and marginal utility that are uncorrelated with aggregate consumption are also uncorrelated with the returns on all assets. Thus, an asset's risk premium, which is determined by the covariance of its return with individuals' marginal utilities of consumption, is unaffected by the fluctuations in individuals' consumption rates that are unrelated to aggregate consumption, because those fluctuations are also unrelated to all asset returns. The reason that asset betas with

respect to only aggregate consumption are in the intertemporal CAPM is that the assets available have betas equal to zero when measured relative to the components of individuals' consumption risks that are uncorrelated with aggregate consumption.

## 6. Asset pricing with no riskless asset

This section derives for an economy with no riskless asset a 'zero-beta' intertemporal CAPM that corresponds to the zero-beta CAPM derived by Lintner (1969), Black (1972) and Vasicek (1971) in a single-period model. The differences between the models are: (1) an asset's beta is measured relative to aggregate consumption, rather than relative to aggregate wealth, and (2) the zero-beta portfolio, whose expected return replaces that of the riskless return in (19), is a portfolio with returns uncorrelated with aggregate consumption, rather than a portfolio with returns uncorrelated with the market portfolio's return.

The only formal modification to the individual's optimization problem [eqs. (6)–(8), section 2] is that the expected rate of return on invested wealth, which was  $w'\mu_a + (1 - I'w)r$ , is now simply a weighted average of risky asset returns,  $w'\mu_a$ . The wealth constraint is now that the risky asset portfolio weights sum to unity, which may be enforced by the use of a Lagrange multiplier in (8). The first-order condition that the marginal utility of consumption equals the marginal utility of wealth is unchanged; however, the optimal risky asset portfolio of (10) now becomes

$$w^k W^k = (-J_w^k / J_{ww}^k) V_{aa}^{-1} \mu_a + (\lambda^k / W^k J_{ww}^k) V_{aa}^{-1} \mathbf{1} - V_{aa}^{-1} V_{as} (J_{sw}^k / J_{ww}^k), \quad (10')$$

where  $\lambda^k$  is individual  $k$ 's Lagrange multiplier for his budget constraint.

By an extension of the proof in appendix 1 for the  $(S+2)$ -fund theorem of section 2, it is seen that an  $(S+2)$ -fund theorem holds in this economy with no riskless asset. The funds may be chosen to be (1) the  $S$  portfolios having the highest correlations, respectively, with the  $S$  state variables summarizing investment opportunities, (2) the market portfolio, and (3) the zero consumption-beta portfolio of the risky assets that has minimum variance.

Substituting partial derivatives of individual  $k$ 's direct utility function for consumption and  $k$ 's optimal consumption function for the partials of the indirect utility function in (10'), and proceeding as in eqs. (12)–(16) in section 3, gives

$$V_{ac_k} = T^k \mu_a + (\lambda^k \mathbf{1} / W^k U_{cc}^k). \quad (16')$$

Aggregating (16') over all individuals gives

$$V_{ac} = T^M \mu_a + v1, \quad (17'')$$

where

$$v = \sum_k \lambda^k / W^k U_{cc}^k.$$

For any portfolio  $z$  with returns that are uncorrelated with aggregate consumption, from (17'') above,

$$v = -T^M \mu_z. \quad (17''')$$

Substituting this into (17'') gives

$$(\mu_M - \mu_z) / \sigma_{MC} = (T^M)^{-1}, \quad (18')$$

and

$$\mu_a - \mu_z 1 = (\beta_{ac} / \beta_{MC}) (\mu_M - \mu_z). \quad (19')$$

Thus, if there is no riskless asset in the single-good continuous-time model, then the equilibrium expected return on an asset is equal to the expected return on a portfolio with returns uncorrelated with aggregate consumption plus a risk premium proportional to the asset's consumption-beta. Both the mutual fund theorem of section 2 and the intertemporal CAPM of section 3 hold with no riskless asset when 'the riskless asset' in those results is replaced by 'the zero consumption-beta portfolio that has minimum variance'.<sup>14</sup>

The next section examines asset pricing in the multi-good continuous-time model, when a nominally riskless asset is assumed to exist.

## 7. Asset pricing with many consumption-goods

The derivations of the consumption, portfolio and pricing results thus far have been in the context of a rather general *single-good* economy. This section discusses some modifications of the results that would occur in a multi-good economy. The major focus will be on conditions that permit the derivation of a 'single-beta' intertemporal capital asset pricing model with stochastic investment *and* consumption opportunities (similar to that of section 3, which had only stochastic investment opportunities).

<sup>14</sup>The choice of a zero-beta, minimum-variance portfolio is intuitive, but not unique. For example, in the mutual fund theorem of section 2, the unconstrained minimum variance portfolio can also replace the riskless asset under the assumptions of this section. Also, the zero-beta intertemporal CAPM can be written in terms of the expected return on any zero consumption-beta portfolio

Let there be  $Q$  goods in the economy and let  $q^k(t)$  be individual  $k$ 's  $Q \times 1$  vector of the rates at which quantities are consumed of the various goods at time  $t$ . Each individual is assumed to maximize the expected utility of a time-additive utility function as in (6), but with  $U^k(c^k, t)$  being replaced by  $u^k(q^k, t)$ . The vector of consumption-goods prices is  $P_c$ , and individual  $k$ 's rate of nominal expenditures is  $c^k = P'_c q^k$ . Individual  $k$ 's indirect utility function for consumption expenditures is now defined as

$$U^k(c^k, P'_c, t) = \max_{\{P_c: q^k = c^k\}} u^k(q^k, t). \quad (26)$$

The analysis of section 2 is virtually unchanged in the multi-good model; first-order conditions (9) and (10) still hold when the state vector  $s$  is assumed to include as a subset  $P_c$  and its probability distribution. The  $(S + 2)$ -fund theorem obtains with instantaneous-maturity commodity futures contracts being perfect hedges for changes in consumption-goods prices. Similarly, the multi-beta asset-pricing model given by (11) holds in this model, with expected excess returns on those futures contracts (if they exist) being a subset of  $(\mu_s^* - r)$ .<sup>15</sup>

Although, in the multi-good case, the form of the demand equations for assets is unchanged from section 2's eq. (10), section 3's translation of those demands in terms of the individual's optimal consumption function is somewhat different in the multi-good case. The difference arises from the fact that the utility of a given level of consumption expenditure now depends upon relative prices,  $P_c$ . Mathematically, in section 2,  $U_c^k(c^k, t) = J_{W^k}^k(W^k, s, t)$  implied that  $U_{cc}^k c_s^k = J_{W^k, s}^k$ , but in the multi-good case we have:  $U_c^k(c^k, P_c(s), t) = J_{W^k}^k(W^k, s, t)$  implies that  $U_{cc}^k c_P^k + U_{cP}^k = J_{W^k, P}^k$  by the implicit function theorem. Thus, individual  $k$ 's asset demand functions, written in terms of his optimal consumption function, are [from (10) and above, assuming the first  $Q$  state variables are the logarithms of consumption-goods prices]

$$\begin{aligned} w^k W^k &= (T^k / c_W^k) V_{aa}^{-1} (\mu_a - r) - V_{aa}^{-1} V_{as} (c_s^k / c_W^k) \\ &\quad - V_{aa}^{-1} V_{as} \begin{pmatrix} U_{\ln P_c, c}^k / U_{cc}^k c_W^k \\ 0 \end{pmatrix} \end{aligned} \quad (27)$$

The last term in (27) represents long or short components of asset demands for the portfolios that are most highly correlated with the prices of consumption-goods; this term arises from the dependence of the individual's

<sup>15</sup>Since futures contracts require no investment and, therefore, rates of return are undefined, the expected excess return on a contract in this context should be viewed as the expected rate of return to a portfolio of the futures contract and an instantaneously riskless bond that has face value equal to the price of the futures contract.

indirect marginal utility for nominal expenditure on consumption-goods prices.

Let  $\alpha^k$  be the  $Q \times 1$  vector of individual  $k$ 's budget shares, i.e.,  $\alpha_j^k = P_j q_j^k / c^k$ , and let  $m^k$  be individual  $k$ 's vector of incremental ('marginal') budget shares, i.e.,  $m_j^k = P_j (\partial q_j^k / \partial c^k)$ . The vector  $m^k$  is the set of fractions of an additional dollar of total expenditure that would be spent on the various consumption-goods. The new term in (27) due to the multi-good model may be expressed in terms of the average and marginal vectors of budget shares as shown in appendix 3, giving asset demands

$$w^k W^k = (T^k / c_W^k) V_{aa}^{-1} (\mu_a - r) - V_{aa}^{-1} V_{as} (c_s^k / c_W^k) + V_{aa}^{-1} V_{as} \begin{pmatrix} c^k \alpha^k / c_W^k - (T^k / c_W^k) m^k \\ 0 \end{pmatrix}. \tag{28}$$

Multiplying (28) by  $(V_{aa} c_W^k)$  and rearranging terms gives

$$T^k \begin{bmatrix} \mu_a - r - V_{as} \begin{pmatrix} m^k \\ 0 \end{pmatrix} \\ \end{bmatrix} = V_{aW_k} c_W^k + V_{as} c_s^k - V_{as} \begin{pmatrix} \alpha^k c^k \\ 0 \end{pmatrix} = V_{ac_k} - V_{as} \begin{pmatrix} \alpha^k c^k \\ 0 \end{pmatrix}, \tag{29}$$

where the second line recognizes that  $c^k = c^k(W^k, s, t)$  and Ito's Lemma implies that  $V_{aW_k} c_W^k + V_{as} c_s^k = V_{ac_k}$ .

Aggregating the optimality condition in (29) for all individuals gives a similar relation in terms of aggregate consumption and aggregate vectors of average budget shares and marginal budget shares,

$$\mu_a - r - V_{as} \begin{pmatrix} m \\ 0 \end{pmatrix} = (T^M / C)^{-1} \left[ V_{a, \ln C} - V_{as} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right], \tag{30}$$

where

$$\alpha = \left( \sum_k \alpha^k c^k \right) / C \quad \text{and} \quad m = \left( \sum_k m^k T^k \right) / T^M.$$

The calculation of the economy-wide vector of average budget shares,  $\alpha$ , requires only data on the aggregate dollars spent on the various goods; no other preference information is required. These shares are the fractions of aggregate expenditure that are spent on the various consumption-goods. These budget shares are, in principal, the weights used in the computation of the price deflator for consumption expenditures in the National Income and Product Accounts.<sup>16</sup>

<sup>16</sup>See the *Survey of Current Business* of the U.S. Department of Commerce.

The vector of aggregate marginal budget shares,  $m$ , is the set of fractions of an additional dollar of aggregate expenditure (allocated optimally among individuals) that would be spent on the various consumption goods, holding prices constant. The reason that this statement can be made, without explicit reference to the risk tolerances of individuals, is that the optimal allocation among individuals of an individual dollar of aggregate nominal expenditure is according to individuals' risk tolerances relative to aggregate risk tolerance,  $(T^k/T^M)$ . Thus, the aggregate marginal budget shares for goods may be written as

$$m_j = \sum_k (\partial c^k / \partial C) P_j (\partial q_j^k / \partial c^k) = P_j (\partial q_j / \partial C). \quad (31)$$

Note that the aggregate marginal budget share for each good can be computed as the product of (1) the aggregate average budget share for the good and (2) the aggregate expenditure elasticity of demand for the good,  $(\partial \ln q_j) / (\partial \ln C)$ .

It is useful for the subsequent analysis to define the local percentage changes in two price indices – one based upon average budget shares for the economy and one based upon the marginal budget shares for the economy,<sup>17</sup>

$$dI/I \equiv \sum_j \alpha_j (dP_j/P_j), \quad dI_m/I_m \equiv \sum_j m_j (dP_j/P_j). \quad (32)$$

The two terms in eq. (30) that involve  $m$  and  $\alpha$  can be rewritten in terms of these price indices, giving

$$\mu_a - r - V_{aI_m} = (T^M/C)^{-1} [V_{a, \ln C} - V_{aI}], \quad (33)$$

where  $V_{aI_m}$  and  $V_{aI}$  are the vectors of the covariances of asset returns with the local percentage changes in the price indices.

Since it can be shown that a feasible (but not necessarily optimal) allocation exists such that everyone in the economy has a consumption

<sup>17</sup>A globally valid price index that is invariant to the level of nominal expenditure exists for an individual if and only if his indifference curves are 'homothetic'. This implies unitary demand elasticities for all goods. When they are not unitary, the individual's budget shares depend upon his level of nominal expenditure, making the weights in his price index vary with the level of expenditure. A survey of price index results is provided by Samuelson and Swamy (1974). Identical and homothetic consumption preferences for all individuals are typically assumed to justify the use of aggregate budget shares to compute a price index for the economy. The price indices used in this paper do not require that individuals be identical, nor that they have homothetic preferences. The continuity of the continuous-time framework and the weaker requirement that the price indices be locally (not globally) valid permits the greater generality of consumption preferences of this paper. For a paper that utilizes preference restrictions that give a globally valid price index, see Grauer and Litzberger (1979).



allocation that is preferable to his current allocation if and only if the percentage change in aggregate nominal expenditure exceeds the percentage change in the average budget share price index,  $I$ , aggregate real consumption is defined as  $C^* = C/I$ .<sup>18</sup> Given this definition, note that the vector of covariances of real asset returns with aggregate real consumption is

$$V_{a \cdot C^*} = V_{a, \ln C} - V_{aI} - V_{IC} \cdot I, \quad (34)$$

where  $V_{IC} \cdot$  is the covariance of the aggregate average-weighted price index with aggregate real consumption. Defining an asset's 'real consumption-beta',  $\beta_j^*$ , as the local covariance of its real return with percentage changes in aggregate real consumption, divided by the variance rate of changes in aggregate real consumption, then (33) can be re-written in terms of assets' real consumption-betas,

$$\mu_a - r - V_{aI_m} = (T^M / C \sigma_{C^*}^2)^{-1} [\beta_a^* - \beta_r^* I], \quad (35)$$

where  $\beta_r^*$  is the real consumption-beta of the nominally riskless asset.

The left-hand side of (35) can be interpreted as the differences of the expected real returns on assets from the expected real return on the nominally riskless asset, where these expected real returns are evaluated relative to the price index with aggregate marginal budget shares. To see this, first note that the instantaneous expected percentage rate of change of an asset's real price,  $P_a/I$ , is given by Ito's Lemma as the expected nominal return on the asset, minus the expected rate of inflation measured by the index, and minus the covariance of the asset's nominal return with inflation. The covariance term is explained by the fact that an asset with high nominal payoffs when prices are low and low nominal payoffs when prices are high buys more real goods on average than an asset with positive covariance of its nominal returns with inflation, assuming that expected nominal payoffs are the same for both assets. Since the covariance of the nominal return on the nominally riskless asset with inflation is zero, the LHS of (35) is the difference between the expected real returns on assets,  $\mu_a^*$ , and the expected real return on the nominally riskless asset,  $\mu_r^*$ . It can be easily verified that for any three assets  $i$ ,  $j$  and  $k$ ,

$$(\mu_i^* - \mu_j^*) / (\beta_i^* - \beta_j^*) = (\mu_k^* - \mu_j^*) / (\beta_k^* - \beta_j^*) \quad \text{for all } i, j, k. \quad (36)$$

Letting  $z$  represent a portfolio with real returns that are uncorrelated with

<sup>18</sup>For a proof of the result stated, see Breeden (1977, ch. 3)

changes in aggregate real consumption, it is seen that a multi-good, zero-beta intertemporal CAPM obtains

$$\mu_i^* - \mu_z^* = (\beta_i^*/\beta_k^*)(\mu_k^* - \mu_z^*). \quad (37)$$

The use of the price index based upon aggregate marginal shares for calculation of expected real returns, while using the price index with aggregate average shares as weights for calculation of real consumption-betas, requires some intuitive explanation. Before proceeding with an explanation, note that there is no difference between the indices if aggregate expenditure elasticities of demand for goods are all unity. This aggregate 'homothetic' case involves strong preference assumptions and is not assumed to hold. As in the single-good economy, asset prices are determined from their payoffs and from individuals' marginal utilities of a dollar of consumption expenditure in the various states of the world. The marginal utility of a dollar to an individual depends upon: (1) the quantities of goods consumed, via diminishing marginal utilities for the consumption of goods, and (2) the quantities of goods that a dollar can buy. By the definition of the marginal budget share vector, an additional dollar is spent on goods in the proportions given by the marginal vector; thus, the price index with marginal weights evaluates the quantities of goods that another dollar purchases. As Samuelson and Swamy (1974) observed, real consumption is a quantity index. As a quantity index, the larger real consumption is, the smaller the marginal utility of goods consumed is. The role of the price index with average budget shares as weights in risk measurement arises from its use in the computation of aggregate real consumption, which is inversely related to the marginal utilities of consumption-goods.

To this point, a real riskless asset is not assumed to exist, nor are futures contracts that can create a real riskless return assumed to exist. If a real riskless asset or portfolio is assumed to exist and have a real return of  $r^*$ , then the expected return on a zero real consumption-beta portfolio in the pricing eq. (37) can be replaced by  $r^*$ ,

$$\mu_i^* - r^* = (\beta_i^*/\beta_k^*)(\mu_k^* - r^*). \quad (38)$$

This is an intertemporal asset pricing model developed in a multi-good world with stochastic consumption and investment opportunities.

The results obtained here may be compared to those obtained in the explicitly multi-commodity economies of Long (1974) and Grauer and Litzenberger (1979). Long makes no restrictions on preferences for goods, but assumes joint normality of consumption-goods prices. The effect of many goods in his model is to extend the number of betas that must be calculated to find the expected excess return on any asset from the expected excess

returns on futures contracts and on portfolios that hedge against investment opportunity set changes. The derivation in this paper of instantaneous expected excess returns in terms of a single beta for each asset is a contribution to the literature.

Grauer and Litzenberger work with a multi-commodity, two-period state preference model and derive asset prices with particular attention to the prices of commodity futures contracts. They make no assumptions about the probability distribution of states of the world, but they assume that the capital markets are Pareto-optimal and that each individual has 'homothetic' preferences for consumption-goods, i.e., that all income elasticities of demand are unity for all goods, for each individual. They derive an asset's risk premium from its return covariance with a single variable, the social marginal utility of wealth. This variable is a function of aggregate wealth deflated by a price index that is assumed to be the same for all individuals. The derivation in this paper of a single-beta measure of risk in a multi-commodity world is similar to theirs, but knowledge of the social marginal utility function is not needed for the beta computation of this paper. The difference between their focus upon wealth and the present focus on consumption is a product of their two-period world, which does not require an analysis of changing investment opportunities. Finally, the preference assumptions needed for the existence of a price index in a discrete-time model are not needed for the local statements of the continuous-time model.

## **8. Conclusion**

An intertemporal capital asset pricing model has been derived in an economic environment permitting both stochastic consumption-goods prices and stochastic portfolio opportunities. The paper is an extension and generalization of Merton's (1973) continuous-time model, deriving equivalent pricing equations that are simpler in form and are potentially empirically testable.

The use of aggregate consumption in empirical tests, rather than the market portfolio that has been used, has both virtues and difficulties. Difficulties with consumption numbers that are available include: (1) instantaneous consumption rates are not measured; rather, weekly, monthly, quarterly, or annual integrals of these rates are measured,<sup>19</sup> (2) only the part of the measured consumption of goods that gives current utility should be included, which excludes a large fraction of current purchases of durables, and (3) the actual data that are available contain considerable measurement error, whereas the prices and numbers of shares used in the market portfolio

<sup>19</sup>With power utility functions and lognormal consumption, this is not a problem, since the pricing model holds with betas and returns measured over any interval. See Breeden and Litzenberger (1978) for a proof of this result.

computations are measured with very little error. The principal virtue of aggregate consumption measures, in comparison with the market proxies used, is that the consumption measures available cover a greater fraction of the true consumption variable than the fraction that the market portfolio measures cover of the true market portfolio (mainly because of the lack of coverage of human capital, real estate, and consumer durables in market measures). Note also that proposed capital expenditure projects typically have cash flows that are more significantly related to aggregate consumption, than to the market portfolio. This may make the distinction of projects with different risk levels more precise and more intuitive, thereby facilitating the use of asset pricing theory in capital budgeting.

In the continuous-time model, areas that need additional theoretical development include the role of firms and their optimal investment and capital structure decisions, and the impact of transaction costs, information costs, and diverse beliefs upon optimal consumption–investment decisions and upon the structure of asset returns.

### Appendix 1: Proof of Theorem 1

By aggregating the optimal portfolio demands of all individuals given by (10) the market portfolio must be

$$\mathbf{w}^M M = \sum_k \mathbf{w}^k W^k = T_W^M \mathbf{V}_{aa}^{-1} (\boldsymbol{\mu}_a - \mathbf{r}) + \mathbf{V}_{aa}^{-1} \mathbf{V}_{as} \mathbf{H}_s^M, \quad (\text{A.1})$$

where

$$T_W^k = -J_W^k / J_{WW}^k \quad \text{and} \quad \mathbf{H}_s^k = -J_{Ws}^k / J_{WW}^k,$$

and where

$$T_W^M = \sum_k T_W^k \quad \text{and} \quad \mathbf{H}_s^M = \sum_k \mathbf{H}_s^k.$$

Substituting (A.1) into (10) allows the individual's portfolio demands to be written as

$$\mathbf{w}^k W^k = (T_W^k / T_W^M) M (\mathbf{w}^M) + \mathbf{V}_{aa}^{-1} \mathbf{V}_{as} (\mathbf{H}_s^k - \mathbf{H}_s^M T_W^k / T_W^M). \quad (\text{A.2})$$

This proves that all individuals may obtain their optimal portfolio positions by trading in  $(S+2)$  'mutual funds', with one of them being the market portfolio, one being the riskless asset, and  $S$  of them being given by  $\mathbf{V}_{aa}^{-1} \mathbf{V}_{as}$ .

Next, note that column  $j$  of  $\mathbf{V}_{aa}^{-1} \mathbf{V}_{as}$ , i.e.,  $\mathbf{V}_{aa}^{-1} \mathbf{V}_{as_j}$ , is the solution to the following problem (up to a factor of proportionality):

$$\frac{1}{2\lambda} \mathbf{V}_{aa}^{-1} \mathbf{V}_{as_j} \quad \text{solves} \quad \max_{\mathbf{w}_j} \{ \mathbf{w}_j' \mathbf{V}_{as_j} + \lambda (\sigma^2 - \mathbf{w}_j' \mathbf{V}_{aa} \mathbf{w}_j) \}. \quad (\text{A.3})$$

In (A.3), by maximizing covariance of the portfolio with  $s_j$  for a given level of variance, we effectively find the portfolio of assets that maximizes the correlation coefficient of its returns with changes in state variable  $j$ . Given this, the theorem is proven from (A.2) and the fact that wealth not in risky assets is placed in the nominally riskless asset. Q.E.D.

**Appendix 2**

Since  $c^k = c^k(W^k, s, t)$  and  $c^j = c^j(W^j, s, t)$ , Ito's Lemma implies that the covariance of  $c^k$  and  $c^j$  is

$$\begin{aligned} \text{cov}(c^k, c^j) &= (c_s^k \mathbf{0} c_s^{k'}) \begin{pmatrix} V_{W^k W^k} & V_{W^k W^j} & V_{W^k s} \\ V_{W^j W^k} & V_{W^j W^j} & V_{W^j s} \\ V_{s W^k} & V_{s W^j} & V_{ss} \end{pmatrix} \begin{pmatrix} 0 \\ c_W^j \\ c_s^j \end{pmatrix}, \\ &= c_s^{k'} V_{s W^j} c_W^j + c_s^{k'} V_{ss} c_s^j + c_W^k V_{W^k s} c_s^j + c_W^k V_{W^k W^j} c_W^j, \end{aligned} \tag{A.4}$$

where subscripted  $V$ 's represent covariance matrices with appropriate dimensions and subscripted  $c$ 's represent partial derivatives or gradients of those consumption functions.

First, the assumption is that

$$V_{aa} = \begin{pmatrix} V_{MM} & V_{Ms} \\ V_{sM} & V_{ss} \end{pmatrix} \equiv V, \tag{A.5}$$

and

$$\mu_a - r = \begin{pmatrix} \mu_M - r \\ \mu_s^* - r \end{pmatrix} \equiv \mu - r. \tag{A.6}$$

Define an  $S \times (S + 1)$  matrix  $L$  to be

$$L = (\mathbf{0} \quad \mathbf{I}), \tag{A.7}$$

where  $\mathbf{0}$  is an  $S \times 1$  vector of zeros and  $\mathbf{I}$  is an  $S \times S$  identity matrix. Note that

$$L(\mu - r) = \mu_s^* - r, \tag{A.8}$$

and

$$LV = V_{s, Ms} \equiv (V_{sM} \quad V_{ss}). \tag{A.9}$$

From individuals' optimal asset demands, (12), it is seen that

$$c_W^k w^k W^k = V^{-1} [T^k (\mu - r) - VL' c_s^k], \quad (\text{A.10})$$

which implies that

$$\begin{aligned} c_W^k V_{sW_k} &= LV [c_W^k w^k W^k] \\ &= T^k L (\mu - r) - LVL' c_s^k. \end{aligned} \quad (\text{A.11})$$

Next, evaluate the last term of (A.4),

$$\begin{aligned} c_W^k V_{W_k W_j} c_W^j &= c_W^k W^k w^{k'} V w^j W^j c_W^j \\ &= [T^k (\mu - r)' - c_s^{k'} LV] V^{-1} [T^j (\mu - r) - VL' c_s^j] \\ &= T^k T^j (\mu - r)' V^{-1} (\mu - r) - T^k (\mu - r)' L' c_s^j \\ &\quad - T^j c_s^{k'} L (\mu - r) + c_s^{k'} LVL' c_s^j. \end{aligned} \quad (\text{A.12})$$

Substituting the results of (A.11) and (A.12) into (A.4) gives the covariance of changes in individual  $k$ 's optimal consumption rate with changes in  $j$ 's optimal consumption rate,

$$\begin{aligned} \text{cov}(c^k, c^j) &= c_s^{k'} [T^j L (\mu - r) - LVL' c_s^j] + c_s^{k'} LVL' c_s^j \\ &\quad + [T^k (\mu - r)' L' - c_s^{k'} LVL'] c_s^j \\ &\quad + T^k T^j \gamma - T^k (\mu - r)' L' c_s^j \\ &\quad - T^j c_s^{k'} L (\mu - r) + c_s^{k'} LVL' c_s^j \\ &= T^k T^j \gamma, \end{aligned} \quad (\text{A.13})$$

where  $\gamma = (\mu - r)' V^{-1} (\mu - r)$ . Eq. (A.13) is eq. (24) of the text, as was to be shown.

### Appendix 3

The definition of the consumer's indirect utility function is

$$U(c, t, P_c) = \max_{P_c, q=c} u(q, t) = \max_q \{u(q, t) + \lambda(c - P_c q)\} \quad (\text{A.14})$$

and the first order conditions for a maximum imply that

$$u_q = \lambda P_c, \quad U_p = -\lambda q, \quad (\text{A.15})$$

and the shadow price  $\lambda = U_c$ .

By differentiating the optimality conditions in (A.15),

$$-U_{\ln P, c}/U_{cc} = (U_c/U_{cc})P_j(\partial q_j/\partial c) + P_j q_j = -Tm_j + \alpha_j. \quad (\text{A.16})$$

Substitute (A.16) into (21), and (22) is obtained.

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